Integration

notations: $f : [a,b] \to \mathbb{R}$ bounded

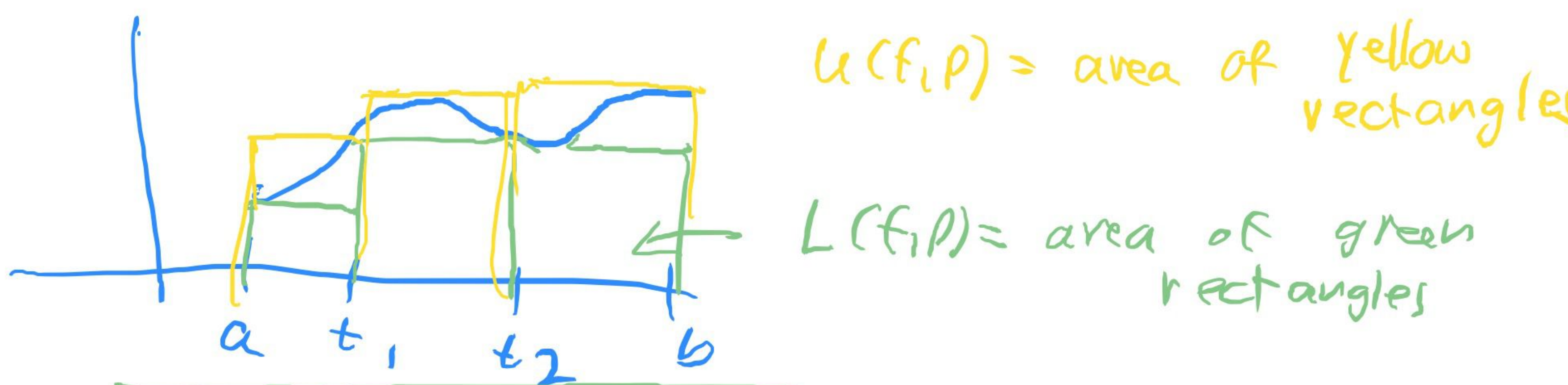
$m(f, [r,s]) = \inf \{ f(x), r \le x \le s \}$

$M(f, [r,s]) = \sup \{ \quad " \quad \quad " \quad \}$

$P = \{ t_0 = a < t_1 < \dots < t_n = b \}$ a partition

Lower Darboux sum: $L(f,P) = \sum_{i=1}^{n} m(f, [t_{i-1}, t_i]) (t_i - t_{i-1})$

upper " " $U(f,P) = \sum_{i=1}^{n} M(f, [t_{i-1}, t_i]) (t_i - t_{i-1})$



Obviously: $\boxed{L(f,P) \le U(f,P)}$

Less obvious:

$$L(f,P) \leq U(f,Q) \quad \text{for } \underline{\text{any}} \text{ partitions } P \text{ and } Q$$

to prove this, we need:

<u>Lemma</u> (32.2 in book)

Assume $P \subset Q$

$$\Rightarrow \quad L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$$

Proof. already seen: middle inequality is clear

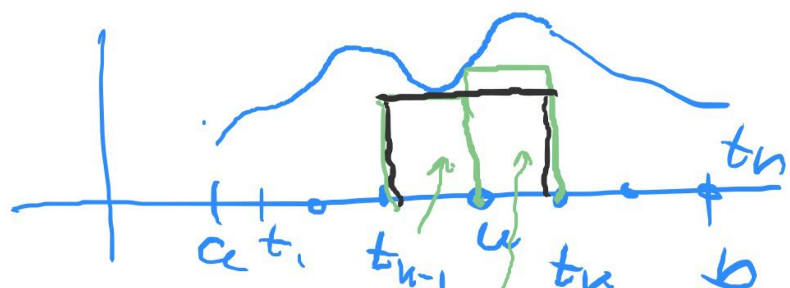
we prove first inequality (proof for 3rd inequality same) step 1 Assume $Q = P$ + one more point u

$$Q = \underbrace{\{t_0, t_1, \dots, t_n\}}_{P} \cup \{u\}$$

to show: $L(f,P) \le L(f,Q)$!

observe: the summands for $L(f,P)$ and $L(f,Q)$ are the same except for

all other rectangles are the same for $L(f,P)$ and for $L(f,Q)$



$\Rightarrow L(f,Q) - L(f,P)$

= area of 2 green rectangles

− " " black rectangle

2 rectangles for $L(f,Q)$

1 rectangle for $L(f,P)$

(all other rectangles cancel)

area of

$$= \underbrace{m(f,[t_{k-1},u])}_{\ge m(f,[t_{k-1},t_k])}(u-t_{k-1}) + m(f,[u,t_k])(t_k-u) - m(f,[t_{k-1},t_k])(t_k-t_{k-1})$$

obviously: minimum over a bigger set $\leq$ min. over a smaller set

$$\Rightarrow \quad m(f, [t_{k-1}, u]) \geq m(f, [t_{k-1}, t_k])$$

$$m(f, [u, t_k]) \geq \quad ''$$

apply these inequalities to expression on bottom of last page

$$\geq m(f, [t_{k-1}, t_k])(u - t_{k-1}) + m(f, [t_{k-1}, t_k])(t_k - u)$$

$$- \; m(f, [t_{k-1}, t_k])(t_k - t_{k-1})$$

$$= m(f, [t_{k-1}, t_k]) \Big[\underbrace{\cancel{u} - t_{k-1} + t_k - \cancel{u} - (t_k - t_{k-1})}_{=0} \Big] \quad \checkmark$$

<u>step 2</u> $\quad Q = P \cup \{u_1, u_2, \ldots, u_r\}$

Define $\quad Q_i = P \cup \{u_1, u_2, \ldots, u_i\}$

$\Rightarrow \quad Q_{i+1} = Q_i \cup \{u_{i+1}\}$

By step 1: $\underset{Q_0}{\underset{\parallel}{L(f,P)}} \leq L(f, Q_1) \leq L(f, Q_2) \leq \ldots \leq L(f, Q_r) = L(f, Q)$

$Q_r = Q.$

Lemma 2 let P, R be any partitions in $[a,b]$

$$\Rightarrow L(f,P) \leq U(f,R)$$

Proof. let $Q = P \cup R$ $\Rightarrow R \subset Q$, $P \subset Q$

By previous lemma.

$$L(f,P) \underset{P \subset Q}{\leq} L(f,Q) \leq U(f,Q) \underset{R \subset Q}{\leq} U(f,R)$$

Theorem: $L(f) \leq U(f)$ for any bounded function f

Proof. pick a partition Q

$\Rightarrow$ $L(f,P) \leq U(f,Q)$ for any partition P

lemma 2

$\Rightarrow$ $L(f) = \sup\{L(f,P),\ P \text{ a partition}\} \leq U(f,Q)$

This is true for <u>any</u> partition Q

$\Rightarrow$ $L(f) \leq \inf\{U(f,Q),\ Q \text{ a partition}\} = U(f)$

Theorem (Cauchy criteria for integrability)

$f : [a,b] \to \mathbb{R}$ bounded.

f integrable $\Longleftrightarrow$ $\forall \varepsilon > 0$ $\exists$ partition P s.t.

$$U(f,P) - L(f,P) < \varepsilon$$

Proof. "$\Rightarrow$" pick $\varepsilon > 0$

f integrable $\Leftrightarrow$ $L(f) = U(f)$

$\exists$ P_1 s.t. $|L(f) - L(f,P_1)| < \varepsilon/2$

& $\exists$ P_2 s.t. $|U(f) - U(f,P_2)| < \varepsilon/2$

Let $P = P_1 \cup P_2$

$$\Rightarrow |U(f,P) - L(f,P)| = |U(f,P) - U(f) + L(f) - L(f,P)|$$

$$\leq |U(f,P) - U(f)| + |L(f) - L(f,P)|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

"$\Leftarrow$" to show: $L(f) = U(f)$

here: by contradiction:

assume f not integrable $\Leftrightarrow$ $L(f) - U(f) = \varepsilon > 0$

$\Rightarrow$ for <u>any</u> partition P we have

$$U(f,P) - L(f,P) \geq U(f) - L(f) = \varepsilon$$

(because $U(f,P) \geq U(f)$
$L(f,P) \leq L(f)$)

$\Rightarrow$ cond. on r.h.s. not satisfied.

<u>Remark:</u> This approach to integration is due to Darboux

One can show (done in book) that this approach is equivalent to the original approach by Riemann